Self-consistent perturbation theory for random matrix ensembles: II

1993 J. Phys. A: Math. Gen. 26 6541

(http://iopscience.iop.org/0305-4470/26/22/046)

J. Phys. A: Math. Gen. 26 (1993) 6541–6547. Printed in the UK

Self-consistent perturbation theory for random matrix ensembles: II

F Leyvraz

Instituto de Física, Laboratorio de Cuernavaca, University of Mexico (UNAM), Apdo postal 20-364, 01000 México DF, Mexico

Received 7 June 1993

Abstract. A method to evaluate perturbations of arbitrary spectra by one of the classical ensembles was presented in a previous paper. Its application to non-trivial problems is cumbersome and I shall show that by combining the previous results with singular perturbation theory, more complicated problems can be tackled. A scaling argument is also presented to obtain a general result that goes beyond the linear repulsion regime that we discussed previously. This result is of considerable interest as it allows us to obtain a good idea of the correlation function if we, in addition, know its long-range behaviour, which can be obtained by straightforward perturbation theory.

1. Introduction

In our previous work [1], hereafter referred to as I, we derived a general relation valid for the two-point function of an arbitrary ensemble (subject only to a certain invariance condition) perturbed by one of the three classical ensembles [2, 3] (GOE, GUE or GSE). This relation is exact but involves the three-point function of the perturbed ensemble, which is unknown in general. It was argued, however, that for weak perturbations it was reasonable to approximate this latter term by the *unperturbed* three-point function. Once one attempts to apply the results of I to a specific problem [4] that goes beyond the simple example treated in I, they turned out to be extremely difficult to work with. This is the case because the perturbation approximations were not exploited fully. In this paper I shall first present a scaling argument for the short-range part of the two-body function that goes considerably beyond the short-range repulsion of levels discussed in I. I shall then proceed to show that the use of singular perturbation theory [5] will give expressions that are easier to handle, and that are actually applied in [4]. Indeed the result of this argument allows us to have a fair idea of the overall two-point function of eigenvalues if we know, in addition, the long-range behaviour of this function, which can be obtained from ordinary perturbation theory.

2. Scaling properties of the short-range part of the two-point function

In I we derived a general relation valid for the two-point function of an arbitrary ensemble (subject only to a certain invariance condition) perturbed by one of the three classical ensembles [2, 3] (GOE, GUE or GSE). This relation is exact but involves the three-point function of the perturbed ensemble, which is unknown in general. It was argued, however,

0305-4470/93/226541+07$07.50

that for weak perturbations it was reasonable to approximate this latter term by the *unperturbed* three-point function. This then led to an equation of the form

$$\begin{aligned} \frac{\partial \rho_2(x;\lambda)}{\partial \lambda} &= C\frac{\partial^2 \rho_2}{\partial x^2} - 2\frac{\partial}{\partial x}\left(\frac{\rho_2}{x}\right) + F(x) \\ F(x) &= -2\frac{\partial}{\partial x}\mathcal{P}\int \frac{\rho_3^{(0)}(x-y)}{y}\,\mathrm{d}y \end{aligned} \tag{2.1}$$

where C is a constant depending on the perturbing ensemble, $F(x)$ is a complicated integral involving the unperturbed three-point function and λ measures the strength of the perturbation. The initial condition is, therefore, the unperturbed two-point function. As it stands, this equation is extremely difficult to work with, due to the presence both of intractable functions as well as singular integrals. For this reason, we were not able to exploit the solution in quadratures presented in I to solve the problem of N GOEs mixed by a weak GOE. Using the methods developed in this paper, on the other hand, the problem admits a solution.

In this paper, I wish to keep the discussion entirely general. The solution to the specific problem that motivated this work is presented in [4]. For this reason, I shall concentrate in the following on the equation:

$$\begin{aligned} \frac{\partial \phi(x;\lambda)}{\partial \lambda} &= C\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial}{\partial x}\left(\frac{\phi}{x}\right) + F(x) \\ \phi(x;0) &= \sigma(x) \end{aligned} \tag{2.2}$$

for small values of λ. In this equation ϕ denotes an arbitrary function and $F(x)$ is a completely general inhomogeneity. The initial condition $\sigma(x)$ will also be taken to be entirely general. Both, however, will be assumed in the main body of the text to be sufficiently smooth near the origin. This assumption is fulfilled in a large number of practical cases. The possibility that $F(x)$ has a logarithmic singularity at zero (which occurs in problems involving the perturbation of N GOEs) is treated in an appendix.

The first and most obvious remark is that the problem has a somewhat singular nature close to $x = 0$. This is seen by noting that the general solution can be found using an eigenfunction expansion, as indicated in I. From this it follows immediately that the solution must vanish as $x^{2/C}$ as $x \to 0$. Since this is not, in general, true for the initial data $\sigma(x)$, the change in the appearance of $\phi(x;\lambda)$ will be particularly strong in the vicinity of $x = 0$. In the following, I will apply standard singular perturbation techniques to this problem, as described, for example, in [5]. One therefore introduces the rescaled variable ξ equal to $x/\sqrt{\lambda}$. Equation (2.2) then becomes

$$\frac{\partial \phi}{\partial \lambda} = \frac{C}{\lambda}\frac{\partial^2 \phi}{\partial \xi^2} - \frac{2}{\lambda}\frac{\partial}{\partial \xi}\left(\frac{\phi}{\xi}\right) + \frac{\xi}{2\lambda}\frac{\partial \phi}{\partial \xi} + F(\sqrt{\lambda}\xi). \tag{2.3}$$

Thus, for small λ, the terms in $1/\lambda$ must first be set equal to zero. This implies that in a first approximation

$$\begin{aligned} &C\frac{\partial^2 \phi_0}{\partial \xi^2} - 2\frac{\partial}{\partial \xi}\left(\frac{\phi_0}{\xi}\right) + \frac{\xi}{2}\frac{\partial \phi_0}{\partial \xi} = 0 \\ &\phi_0(\xi) \propto \xi^{2/C} \qquad (\xi \to 0) \\ &\lim_{\xi\to\infty} \phi_0(\xi) = \sigma(0). \end{aligned} \tag{2.4}$$

Note that the initial conditions are determined by the following: the condition at $\xi = 0$ is the only one that is compatible with the known small-x properties of the solution described in I. As to the condition for $\xi \to \infty$, it is determined by the fact that $\phi(x;\lambda)$ should not differ much from $\sigma(x)$ for $x \to 0$.

Equation (2.4) is readily solved to yield

$$\phi_0(\xi) = \sigma(0)\frac{\Gamma(1/2)}{(4C)^k\Gamma(1+2k)}\xi^{2k}M_{k,k}(\xi^2/4C)\mathrm{e}^{-\xi^2/8C}$$
$$k = \frac{1}{2}\left(\frac{1}{C}-\frac{1}{2}\right). \tag{2.5}$$

where $M_{k,m}(x)$ denotes the confluent hypergeometric function in the notation of Whittaker and Watson [6]. These functions have already been found in I for the two-point function of a Poisson ensemble perturbed by a weak GOE (though they were given in integral form and not properly identified as confluent hypergeometric functions). It is now clear that their significance is far deeper: they represent the behaviour close to $x = 0$ of any solution of equation (2.2) for small λ. Thus the two-point function of *any* ensemble weakly perturbed by a GOE is determined at small x by the probability $\sigma(0)$ of finding two nearby eigenvalues in the unperturbed ensemble. Furthermore, the way in which the two-point function rises from zero to $\sigma(0)$ is a universal function of $x/\sqrt{\lambda}$ of the form given in equation (2.5). Note that the asymptotic behaviour of the confluent hypergeometric function is well known (see, for example, [6]) and leads to the following large-ξ behaviour for $\phi_0(\xi)$

$$\phi_0(\xi) \approx \sigma(0)(1+2\xi^{-2}). \tag{2.6}$$

In this section, a universal small-x behaviour of the two-point function in terms of the scaling variable ξ has been found. In the next section we shall try to connect this solution to the long-range behaviour expressed in terms of ordinary perturbation theory. Since the former is universal, whereas the latter depends on the details of the perturbation, it is clear that the two solutions will not automatically match. The computation of higher terms in λ will be shown to allow matching of both solutions.

3. Matching the long- and short-range solutions

The next problem is to determine the solution of equation (2.2) in the case of $x \gg \sqrt{\lambda}$. In this case, ordinary perturbation theory is sufficient. I assume that $\phi(x;\lambda)$ will be only slightly different from its initial value $\sigma(x)$. Thus, introducing $\rho(x;\lambda) = \phi(x;\lambda) - \sigma(x)$, one finds

$$\frac{\partial\rho(x;\lambda)}{\partial\lambda} = C\frac{\partial^2\rho}{\partial x^2} - 2\frac{\partial}{\partial x}\left(\frac{\rho}{x}\right) + G(x)$$
$$\rho(x;0) = 0 \tag{3.1}$$
$$G(x) = F(x) + C\frac{\partial^2\sigma}{\partial x^2} - 2\frac{\partial}{\partial x}\left(\frac{\sigma(x)}{x}\right).$$

If $\rho(x;\lambda)$ goes to zero as $\lambda \to 0$, then for small λ, the right-hand side of equation (3.1) is dominated by $G(x)$, from which it follows in a first approximation that

$$\phi(x;\lambda) = \sigma(x) + \lambda G(x) + \mathrm{O}(\lambda). \tag{3.2}$$

In order to see whether this can indeed be made to fit with the small-x (or large-ξ) behaviour discussed above, one needs to compute higher-order terms to the latter. In the ξ variables, the following asymptotic form for $\phi(\xi;\lambda)$ is expected:

$$\phi(\xi;\lambda) = \phi_0(\xi) + \sqrt{\lambda}\phi_1(\xi) + \lambda\phi_2(\xi) + O(\lambda). \tag{3.3}$$

Intuitively, this can be seen as follows: if $\sigma'(0) \neq 0$, then one expects $\phi'(\xi;\lambda)$ to go as $\sqrt{\lambda}$ for $\xi \to \infty$. From this follows the need for a correction of order $\sqrt{\lambda}$. As to the correction of order λ, it is necessary in order to have any hope of matching the small- and large-x solutions.

Putting the definition of $\phi_1(\xi)$ given by equation (3.3) in the equation (2.3) for $\phi(\xi;\lambda)$, I obtain

$$\begin{aligned} &C\phi_1'' + \frac{1}{2}(\xi\phi_1' - \phi_1) - 2\left(\frac{\phi_1}{\xi}\right)' = 0 \\ &\phi_1(\xi) \propto \xi^{2/C} \qquad (\xi \to 0) \\ &\phi_1'(\xi) \to \sigma'(0) \qquad (\xi \to \infty). \end{aligned} \tag{3.4}$$

The first boundary condition is, again, imposed by the condition that $\phi_1(\xi)$ be regular at the origin, whereas the second follows from the above estimate of the derivative of $\phi(\xi;\lambda)$. From this one immediately finds the solution

$$\phi_1(\xi) = \sigma'(0)\frac{2\xi}{(4C)^{(1/C)-(1/2)}\Gamma[(1/C)-(1/2)]}\int_0^\xi \mathrm{d}y\, y^{(2/C)-2}\mathrm{e}^{-y^2/4C} \tag{3.5}$$

as long as $C < 2$. If $C = 2$, which happens in the GOE case, the function $\sigma'(0)\xi$ is a solution satisfying all boundary conditions.

In order to calculate the next order, I must make some assumptions about the behaviour of $F(x)$ near the origin. In the following, I will assume it remains bounded. I will later investigate the consequences of a logarithmic singularity. Both of these cases do indeed occur in practical applications, as set out in [4]. Putting the ansatz for $\phi(x;\lambda)$ given by equation (3.3) into equation (2.3), one finds

$$C\phi_2'' + \frac{\xi}{2}\phi_2' - \phi_2 - 2\left(\frac{\phi_2}{\xi}\right)' + F(0) = 0. \tag{3.6}$$

In order to fix the boundary conditions, note that

$$\begin{aligned} \phi(\xi \to \infty;\lambda) = \phi(x \to 0;\lambda) &= \sigma(0) + \sigma'(0)x + \tfrac{1}{2}\sigma''(0)x^2 + \lambda G(0) \\ &\approx \sigma(0) + \sqrt{\lambda}\sigma'(0)\xi + \lambda(\tfrac{1}{2}\sigma''(0)\xi^2 + G(0)). \end{aligned} \tag{3.7}$$

One may therefore impose the boundary conditions

$$\begin{aligned} &\phi_2(x) \propto x^{2/C} \qquad (x \to 0) \\ &\phi_2(\xi)/\xi^2 \to \sigma''(0)/2 \qquad (\xi \to \infty). \end{aligned} \tag{3.8}$$

The homogeneous equation corresponding to equation (3.6) has the solution

$$\begin{aligned} &\psi_1(\xi) = \frac{\Gamma(3/2)}{(4C)^k\Gamma(3+2k)}\xi^{2k+2}M_{k,k+1}(\xi^2/4C)\mathrm{e}^{-\xi^2/8C} \\ &k = \frac{1}{2}\left(\frac{1}{C} - \frac{5}{2}\right). \end{aligned} \tag{3.9}$$

Using standard formulae for the large-ξ behaviour of the confluent hypergeometric functions [6], one finds that this function goes as $\xi^2 +$ constant. Further, it is seen by inspection that

$$\psi_0(\xi) = \frac{-F(0)}{2(C-1)}\xi^2 \tag{3.10}$$

is a particular solution of equation (3.6). Thus, the boundary conditions can readily be met by adding a suitable multiple of $\psi_1(\xi)$ to $\psi_0(\xi)$. Upon noting further that any solution of equation (3.6) that tends to a limit for large ξ must tend to the value $F(0)$, the asymptotic behaviour of the solution is found to be

$$\begin{aligned}\phi_2(\xi) &= \psi_0(\xi) + \left(\sigma''(0)/2 + \frac{F(0)}{2(C-1)}\right)\psi_1(\xi)\\ &\approx \tfrac{1}{2}\sigma''(0)\xi^2 + F(0) + (C-1)\sigma''(0).\end{aligned} \tag{3.11}$$

Note that, athough the intermediate steps are illegitimate in the specific case $C = 1$, the final result does have a well-defined limit as $C \to 1$. This will not be written out explicitly, as the case it corresponds to is not the most interesting. On the other hand, it should be pointed out that in this case the solutions simplify appreciably. This may, in part, account for the exact solubility of the case of N GUEs perturbed by a GUE, which has been solved exactly for arbitrary perturbation strength in [7].

Combining all the above results, one finds the following large-ξ behaviour for $\phi(\xi;\lambda)$:

$$\phi(\xi;\lambda) \approx \sigma(0)(1+2/\xi^2) + \sqrt{\lambda}\sigma'(0)\xi + \lambda(\sigma''(0)/2)\xi^2 + \lambda F(0) + (\lambda(C-1))\sigma''(0) \tag{3.12}$$

where the first expression is derived from the asymptotic expression for large ξ for $\phi_0(\xi)$. Let us now look at the behaviour of $\phi(x;\lambda)$ in the regime $\sqrt{\lambda} \ll x \ll 1$. In this case, one finds

$$\begin{aligned}\phi(x;\lambda) &\approx \sigma(x) + \lambda G(x)\\ &\approx \sigma(0) + x\sigma'(0) + \frac{x^2}{2}\sigma''(0) + \lambda\left(F(0) + C\sigma''(0) + 2\frac{\sigma(0)}{x^2} - \sigma''(0)\right)\end{aligned} \tag{3.13}$$

which corresponds term for term to the large-ξ behaviour given in equation (3.12). Thus the two perturbation series coincide in the domain where both should be valid.

As to the case in which $F(x)$ has a logarithmic singularity at the origin, which takes place in the case of a GOE, the calculations become rather involved and are developed in the appendix. Summarizing the results, one finds that the higher-order corrections have the form

$$\phi(\xi;\lambda) = \phi_0(\xi) + \sqrt{\lambda}\phi_1(\xi) + \lambda\ln\lambda\psi_2(\xi) + \lambda\phi_2(\xi). \tag{3.14}$$

The consistency of this form with the small-x limit of the perturbative solution can indeed be shown but the functions $\psi_2(\xi)$ and $\phi_2(\xi)$ cannot be computed explicitly. They can, however, be evaluated numerically.

4. Conclusions

Summarizing, I have developed a general method to describe accurately the solution to equation (2.2) for $\lambda \ll 1$ over the whole range of x. Since this equation was derived in order to describe the two-point function of random matrix ensembles perturbed by a classical ensemble of strength λ when $\lambda \ll 1$ one can indeed obtain the two-point function in this regime. The results are somewhat subtle, as they require singular perturbation theory to describe the abrupt change in the nature of the two-point function near the origin as the perturbation is turned on.

The final results can be summarized by saying that there is a universal function which, at a suitable scale, describes the behaviour near the origin. This function accounts for the sudden decrease in the contact probability from its initial value to zero. Corrections of various orders were then computed, in order to ascertain whether the behaviour at the origin could indeed be matched smoothly to the behaviour at $\sqrt{\lambda} \ll x \ll 1$, in which case ordinary perturbation theory is applicable. This was indeed found to be the case.

As a final remark, the following should be pointed out: in I we claimed that the perturbed two-point function always behaved as $x^{2/C}$, even if the eigenvalue repulsion had originally been stronger. Further, it was conjectured that the strength of this effect would be very low. In particular, if $\sigma(x)$ is taken to vanish identically for sufficiently small x, it was suggested that the effect, although present, would be non-perturbative in λ. These conjectures receive some support from the present work: indeed, it was systematically found that the size of the various corrections depended on the various derivatives of $\sigma(x)$ at $x = 0$. From this can be deduced that the effect shown in I would not be observed, under those circumstances, at least to first order in λ. It can probably readily be shown that no corrections will occur at any order in λ.

Acknowledgments

It is a pleasure for me to acknowledge discussions with T H Seligman, without which this paper would not have been completed. Financial support from DGAPA grant IN100491 is also gratefully acknowledged.

Appendix

In the case where the function $F(x)$ has a logarithmic singularity at the origin, the ansatz for the $\phi(\xi;\lambda)$ is as follows

$$\phi(\xi;\lambda) = \phi_0(\xi) + \sqrt{\lambda}\phi_1(\xi) + \lambda\ln\lambda\psi_2(\xi) + \lambda\phi_2(\xi). \tag{A.1}$$

I will assume that $F(x)$ goes as $\Lambda \ln x/x_0$ for $x \to 0$, with x_0 chosen in such a way that there is no constant correction term to this behaviour. For the functions ψ_2 and ϕ_2 this leads to the following coupled equations:

$$\begin{aligned} C\psi_2'' + \frac{\xi}{2}\psi_2' - \psi_2 - 2\left(\frac{\psi_2}{\xi}\right)' + \frac{\Lambda}{2} &= 0 \\ C\phi_2'' + \frac{\xi}{2}\phi_2' - \phi_2 - 2\left(\frac{\phi_2}{\xi}\right)' - \psi_2 + \Lambda\ln\frac{\xi}{x_0} &= 0. \end{aligned} \tag{A.2}$$

The boundary conditions are again obtained by matching the large-ξ and the small-x behaviour. This leads to

$$\begin{aligned}
&\psi_2(\xi) \to \tfrac{1}{2}\Lambda \qquad (\xi \to \infty) \\
&\phi_2(\xi) \approx \frac{\sigma''(0)}{2}\xi^2 + \Lambda \ln\frac{\xi}{x_0} \qquad (\xi \to \infty)
\end{aligned} \tag{A.3}$$

and both $\psi_2(\xi)$ and $\phi_2(\xi)$ go as $\xi^{2/C}$ as $\xi \to 0$. From this one obtains

$$\begin{aligned}
&\psi_2(\xi) = \frac{\Lambda\xi^2}{4(C-1)}\left(\frac{\Gamma(3/2)}{(4C)^k\Gamma(3+2k)}\xi^{2k}\mathrm{e}^{-\xi^2/8C}M_{k,k+1}\left(\frac{\xi^2}{4C}\right) - 1\right) \\
&k = \frac{1}{2}\left(\frac{1}{C} - \frac{5}{2}\right).
\end{aligned} \tag{A.4}$$

The calculation of $\phi_2(\xi)$, on the other hand, turns out to be impracticable. The following device can be used for its numerical evaluation, however: solve the equation for $\phi_2(\xi)$ numerically using the exact form of $\psi_2(\xi)$ as input and the initial condition

$$\phi_2(0) = 0 \qquad \phi_2'(0) = 1. \tag{A.5}$$

The asymptotic behaviour for $\xi \to \infty$ can then be evaluated and adjusted to the expected behaviour by adding a suitable multiple of the solution of the homogeneous equation, which is given by

$$\begin{aligned}
&\frac{\Gamma(3/2)}{(4C)^k\Gamma(3+2k)}\xi^{2k+2}M_{k,k+1}(\xi^2/4C)\mathrm{e}^{-\xi^2/8C} \\
&k = \frac{1}{2}\left(\frac{1}{C} - \frac{5}{2}\right)
\end{aligned} \tag{A.6}$$

where the prefactor has been chosen so as to yield an asymptotic behaviour of ξ^2 as $\xi \to \infty$.

References

[1] Leyvraz F and Seligman T H 1990 *J. Phys. A: Math. Gen.* **23** 1555
[2] Hua L K 1963 *Harmonic Analysis of Functions of Several Complex Variables* transl. L Ebner and A Korány (Providence, RI: American Mathematical Society)
[3] Brody T A *et al* 1981 *Rev. Mod. Phys.* **53** 385
[4] Leyvraz F and Seligman T H 1993 submitted for publication
[5] see e.g., Hinch E J 1991 *Perturbation Methods* (Cambridge: Cambridge University Press)
[6] Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)
[7] Guhr T and Weidenmüller H A 1991 *Ann. Phys.* **199** 417